Analytical Solution for Planar Librations of a Gravity-Stabilized Satellite

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I. Introduction

ATTITUDE stabilization of artificial satellites using gravity-gradient torque has been a subject of extensive investigation. Beletskii[1] reviewed and studied various aspects of planar librations in an eccentric orbit, both analytically and numerically. Modi and Brereton[2] analyzed the problem through the WKBJ approach. Using the method of harmonic balance and numerical integration,[3] they also studied the stability of planar periodic solutions. Their results overlap those in Ref. 1. This Note makes a modest attempt to derive the analytic response through the two-variable expansion procedure.[4] From the solution, a closed form relation among the initial conditions, eccentricity of the orbit and inertia parameter is also derived for the periodic solution at orbital frequency. In the linear range of pitching librations, a good agreement is exhibited between the analytic and numerical solutions.

II. Analysis

For an arbitrary shaped, gravity-stabilized, rigid satellite in an elliptic orbit around a spherical planet, the pitch (ψ) equation of motion with true anomaly (θ) as the independent variable, is[2]

$$\psi''(1+eC_\theta)-2eS_\theta(1+\psi')+3K_iS_\psi C_\psi=0 \tag{1}$$

where the primes indicate differentiation with respect to θ, e = orbit eccentricity, K_i = inertia parameter, $(I_x-I_z)/I_y$ with I_x, I_y, I_z denoting roll, pitch, and yaw moments of inertia, respectively, and S_θ, $C_\theta=\sin\theta$, $\cos\theta$.
Linearization transforms the equations to

$$\psi''(1+eC_\theta)-2eS_\theta(1+\psi')+3K_i\psi=0 \tag{2}$$

To obtain a uniformly valid asymptotic solution of Eq. (2), a fast variable θ_0, a slow variable θ_1, and the dependent variable $\psi(\theta)$ are all assumed to possess the following expansions

$$\theta_0=\theta \qquad \theta_1=\sum_{j=1}^{3}e^j\tau_j\theta+0(e^4)$$

$$\psi(\theta)=\psi(\theta_0,\theta_1)=\sum_{j=0}^{3}e^j\psi_j(\theta_0,\theta_1)+0(e^4) \tag{3}$$

where the coefficients τ_j are to be determined in the course of the analysis. Substituting Eq. (3) into Eq. (2), and comparing coefficients of equal powers of e on both sides gives

$$\text{Order } e^0\text{: } D_{00}\psi_0+\omega_0^2\psi_0=0 \tag{4a}$$

$$\text{Order } e^1\text{: } D_{00}\psi_1+\omega_0^2\psi_1=-2D_{01}\psi_0\psi_0\tau_1-D_{00}\psi_0C_{\theta_0}+2D_0\psi_0S_{\theta_0}+2S_{\theta_0} \tag{4b}$$

$$\text{Order } e^2\text{: } D_{00}\psi_2+\omega_0^2\psi_2=-2D_{01}\psi_0\tau_2-D_{11}\psi_0\tau_1^2-2D_{01}\psi_1\tau_1-2D_{01}\psi_0\tau_1C_{\theta_0}-D_{00}\psi_1C_{\theta_0}+2D_1\psi_0\tau_1S_{\theta_0}+2D_0\psi_1S_{\theta_0} \tag{4c}$$

$$\text{Order } e^3\text{: } D_{00}\psi_3+\omega_0^2\psi_3=-[2D_{01}\psi_0\tau_3+2D_{11}\psi_0\tau_1\tau_2+2D_{01}\psi_1\tau_2+D_{11}\psi_1\tau_1^2+2D_{01}\psi_2\tau_1+C_{\theta_0}(2D_{01}\psi_0\tau_2+D_{11}\psi_0\tau_1^2+2D_{01}\psi_1\tau_1+D_{00}\psi_2)]+2S_{\theta_0}[D_1\psi_0\tau_2+D_1\psi_1\tau_1+D_0\psi_2] \tag{4d}$$

The subscripts 0 and 1 for D denote partial differentiation with respect to θ_0 and θ_1 and $\omega_0^2=3K_i$. The zeroth-order bounded solution from Eq. (4a) is

$$\psi_0(\theta_0,\theta_1)=A_0(\theta_1)S_{\omega_0\theta_0}+B_0(\theta_1)C_{\omega_0\theta_0} \tag{5}$$

where the integration constants are functions of the slow variable. Employing Eq. (5) in (4b) and suppressing resonance terms yields

$$2\tau_1\omega_0\mathrm{d}A_0/\mathrm{d}\theta_1=0 \qquad 2\tau_1\omega_0\mathrm{d}B_0/\mathrm{d}\theta_1=0 \tag{6}$$

which leads to

$$\tau_1=0 \tag{7}$$

and gives the first-order solution as

$$\psi_1(\theta_0,\theta_1)=A_1(\theta_1)S_{\omega_0\theta_0}+B_1(\theta_1)C_{\omega_0\theta_0}+[\omega_0(\omega_0/2-1)/(2\omega_0-1)][A_0S_{(\omega_0-1)\theta_0}+B_0C_{(\omega_0-1)\theta_0}]-[\omega_0(\omega_0/2+1)/(2\omega_0+1)][A_0S_{(\omega_0+1)\theta_0}+B_0C_{(\omega_0+1)\theta_0}]+2S_{\theta_0}/(\omega_0^2-1) \tag{8}$$

To determine the dependence of A_0 and B_0 on θ_1, Eqs. (5,7, and 8) are used in (4c). The solution is rendered uniformly valid by requiring

$$2\tau_2\mathrm{d}A_0/\mathrm{d}\theta_1+B_0\omega_1/2=0 \qquad 2\tau_2\mathrm{d}B_0/\mathrm{d}\theta_1-A_0\omega_1/2=0 \tag{9}$$

whose solution is

$$A_0(\theta_1)=-a\sin(\omega_1\theta_1/4\tau_2-b)$$
$$B_0(\theta_1)=a\cos(\omega_1\theta_1/4\tau_2-b) \tag{10}$$

where a,b are integration constants and

$$\omega_1=3\omega_0(\omega_0^2-1)/(4\omega_0^2-1)$$

Hence, the second-order solution is

$$\psi_2(\theta_0,\theta_1)=A_2(\theta_1)S_{\omega_0\theta_0}+B_2(\theta_1)C_{\omega_0\theta_0}+[\omega_0(\omega_0/2-1)/(2\omega_0-1)][A_1S_{(\omega_0-1)\theta_0}+B_1C_{(\omega_0-1)\theta_0}]-[\omega_0(\omega_0/2+1)/(2\omega_0+1)][A_1S_{(\omega_0+1)\theta_0}+B_1C_{(\omega_0+1)\theta_0}]+3S_{2\theta_0}/(\omega_0^2-1)(\omega_0^2-4)+[\omega_0(\omega_0-3)(\omega_0/2-1)/8(2\omega_0-1)][A_0S_{(\omega_0-2)\theta_0}+B_0C_{(\omega_0-2)\theta_0}]+[\omega_0(\omega_0/2+1)(\omega_0+3)/8(2\omega_0+1)][A_0S_{(\omega_0+2)\theta_0}+B_0C_{(\omega_0+2)\theta_0}] \tag{11}$$

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To determine A_1 and B_1, Eqs. (5,7,8,11) are used in (4d) to give

$$D_{00}\psi_3+\omega_0^2\psi_3=-2\tau_3\omega_0[\mathrm{d}A_0/\mathrm{d}\theta_1.C_{\omega_0\theta_0}-\mathrm{d}B_0/\mathrm{d}\theta_1.S_{\omega_0\theta_0}]$$
$$-2\tau_2\omega_0[\mathrm{d}A_1/\mathrm{d}\theta_1.C_{\omega_0\theta_0}-\mathrm{d}B_1/\mathrm{d}\theta_1.S_{\omega_0\theta_0}]$$
$$+[\omega_0^2(1-7\omega_0^2)/2(4\omega_0^2-1)][A_1S_{\omega_0\theta_0}+B_1C_{\omega_0\theta_0}]$$
$$+[\omega_0^2(2\omega_0^2+1)/(4\omega_0^2-1)][A_1S_{\omega_0\theta_0}+B_1C_{\omega_0\theta_0}]$$
$$+\text{nonsecular terms} \qquad (12)$$

where only the secular terms are gathered to avoid a plethora of terms. Employing Eq. (10) in Eq. (12), the elimination of secular terms gives

$$\mathrm{d}A_1/\mathrm{d}\theta_1+\omega_1B_1/4\tau_2=(a\omega_1\tau_3/4\tau_2^2)\cos(\omega_1\theta_1/4\tau_2-b) \qquad (13a)$$

$$\mathrm{d}B_1/\mathrm{d}\theta_1-\omega_1A_1/4\tau_2=(a\omega_1\tau_3/4\tau_2^2)\sin(\omega_1\theta_1/4\tau_2-b) \qquad (13b)$$

whose general solution is

$$A_1^2+B_1^2=c^2+\theta_1[ac\omega_1\tau_3/2\tau_2^2]\sin(d-b) \qquad (14)$$

where c and d are integration constants. For a uniformly valid solution

$$\tau_3=0 \qquad (15)$$

and thus Eq. (13) has a solution

$$A_1(\theta_1)=-c\sin(\omega_1\theta_1/4\tau_2-d) \qquad (16a)$$

$$B_1(\theta_1)=c\cos(\omega_1\theta_1/4\tau_2-d) \qquad (16b)$$

Adopting Eqs. (7 and 15), the slow variable becomes

$$\theta_1=\tau_2e^2\theta+O(e^4) \qquad (17)$$

Eqs. (10, 16, and 17) modify the zeroth and first-order solutions to

$$\psi_0(\theta)=a\cos(\bar\omega\theta-b) \qquad (18)$$

$$\psi_1(\theta)=c\cos(\bar\omega\theta-d)+[\omega_0(\omega_0/2-1)/(2\omega_0-1)]$$
$$\times a\cos[(\bar\omega-1)\theta-b]-[\omega_0(\omega_0/2+1)/(2\omega_0+1)]$$
$$\times a\cos[(\bar\omega+1)\theta-b]+2S_\theta/(\omega_0^2-1) \qquad (19)$$

respectively, where

$$\bar\omega=\omega_0[1+3e^2(\omega_0^2-1)/4(4\omega_0^2-1)]$$

The independence of $\psi_0(\theta)$ and $\psi_1(\theta)$ from the value of τ_2 should be noticed. However, in view of τ_1 and τ_3 being zero, this independence is expected. Indeed, it is equivalent to choosing another slow variable obtained by dividing $\theta_1(\theta)$ by τ_2 in Eq. (3b). The socalled zeroth-order solution is, in fact, of second order in eccentricity. To determine $A_2(\theta_1)$ and $B_2(\theta_1)$ in the second-order solution Eq. (11), one should consider the differential equation corresponding to the order e^4. However, noting that the functional dependence of A_0 and B_0 in Eq. (10) on the slow variable is identical to that of A_1 and B_1 in Eq. (16), the need to determine the same for A_2 and B_2, too, may be obviated by assuming

$$A_2(\theta_1)=-g\sin(\omega_1\theta_1/4\tau_2-h) \qquad (20a)$$

$$B_2(\theta_1)=g\cos(\omega_1\theta_1/4\tau_2-h) \qquad (20b)$$

where g and h are new constants of integration. The second-order solution Eq. (11) thus modifies to

$$\psi_2(\theta)=g\cos(\bar\omega\theta-h)$$
$$+[\omega_0(\omega_0/2-1)/(2\omega_0-1)]c\cos[(\bar\omega-1)\theta-d]$$
$$-[\omega_0(\omega_0/2+1)/(2\omega_0+1)]c\cos[(\bar\omega+1)\theta-d]$$
$$+[\omega_0(\omega_0+3)(\omega_0/2+1)/8(2\omega_0+1)]a\cos[\bar\omega+2)\theta-b]$$
$$+[\omega_0(\omega_0+3)(\omega_0/2+1)/8(2\omega_0+1)]a\cos[(\bar\omega+2)\theta-b]$$
$$+[3/(\omega_0^2-1)(\omega_0^2-4)]S_{2\theta} \qquad (21)$$

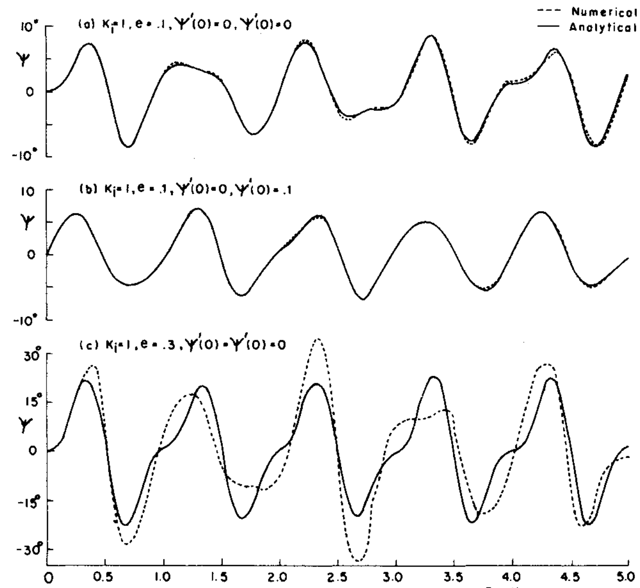

Fig. 1 Comparison between analytical and numerical results.

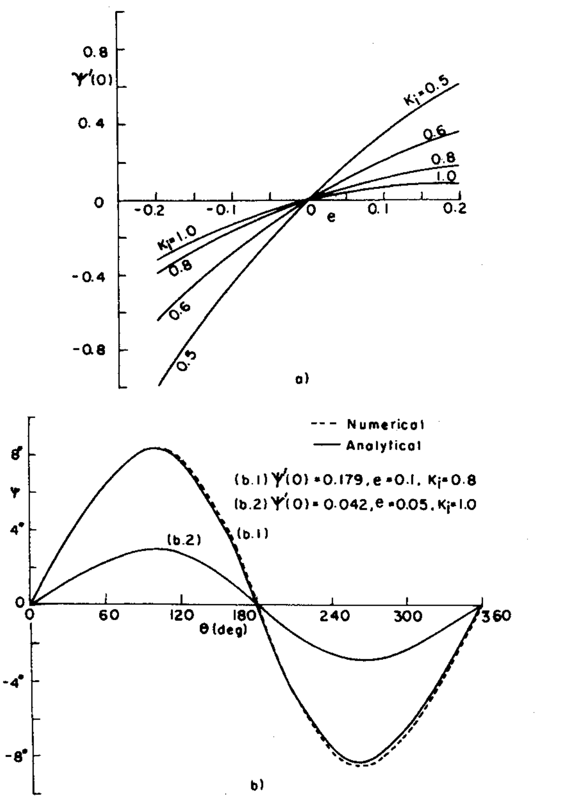

Fig. 2 Periodic solutions at orbital periodicity: a) variation of initial derivatives; b) specific periodic responses.

As the governing differential equation is of the second order, only two integration constants are permitted. Hence it may be assumed that

$$a \equiv c \equiv g \text{ and } b \equiv d \equiv h$$

Thus the final solution up to fourth order in eccentricity is

$$\psi = (1+e+e^2)a\cos(\bar{\omega}\theta - b) + e(1+e)a\omega_0 \left[\frac{\omega_0/2-1}{2\omega_0-1} \cos\{(\bar{\omega}-1)\theta - b\} - \frac{\omega_0/2+1}{2\omega_0+1} \cos\{(\bar{\omega}+1)\theta - b\} \right] + \frac{e^2 a\omega_0}{8} \left[\frac{\omega_0/2-1}{2\omega_0-1} (\omega_0-3)\cos\{(\bar{\omega}-2)\theta - b\} + \frac{(\omega_0/2+1)(\omega_0+3)}{(2\omega_0+1)} \cos\{(\bar{\omega}+2)\theta - b\} \right] + \frac{2e}{\omega_0^2-1}\left[\sin\theta + \frac{3e}{2(\omega_0^2-4)}\sin 2\theta\right] \tag{23}$$

where the constants a and b are correlated to the initial conditions as

$$\psi(0) = a\cos b[(1+e+e^2) + e(1+e)\omega_0\{(\omega_0/2-1)/(2\omega_0-1) - (\omega_0/2+1)/(2\omega_0+1)\} + (e^2\omega_0/8)\{(\omega_0-1)(\omega_0/2-3)/(2\omega_0-1) + (\omega_0/2+1)(\omega_0+3)/(2\omega_0+1)\}] \tag{24a}$$

$$\psi'(0) = a\sin b\{\bar{\omega}(1+e+e^2) + e(1+e)\omega_0[(\omega_0/2-1)(\bar{\omega}-1)/(2\omega_0-1) - (\omega_0/2+1)(\bar{\omega}+1)/(2\omega_0+1)] + (e^2\omega_0/8)[(\omega_0/2-1)(\omega_0-3)(\bar{\omega}-2)/(2\omega_0-1) + (\omega_0/2+1)(\omega_0+3)(\bar{\omega}+2)/(2\omega_0+1)]\} + 2e[1+3e/(\omega_0^2-4]/(\omega_0^2-1) \tag{24b}$$

Periodic Solutions

The importance of periodic responses for librating systems cannot be overemphasized. They form the spines of the integral manifold of the systems.[3,5,6] Earlier these periodic solutions have been determined numerically and by using the method of harmonic balance. The analytical solution found above readily helps in establishing conditions for the periodic motion.

Equation (23) indicates that the response of the system is, in general, a combination of several terms with irrational frequencies. However, if the constants a and b are rendered zero through suitably chosen initial conditions, the response becomes periodic with the orbital frequency. These conditions are

$$\psi(0) = 0 \tag{25a}$$

$$\psi'(0) = [2e/(\omega_0^2-1)][1+3e/(\omega_0^2-4)] \tag{25b}$$

$$\omega_0^2 = 3K_i \tag{25c}$$

and the solution is then represented by the last term of Eq. (23).

III. Discussion of Results

To establish the region of validity for this solution, a comparison was made with numerical integration of the exact Eq. (1). The results were obtained using Adams-Bashforth predictor-corrector in quadrature with Runge-Kutta starter with a step size of $\theta = 3°$ on IBM 360/44 computer. A wide range of satellite inertia, orbital eccentricity and initial conditions were considered. A few typical results are shown in Fig. 1. Fig. 1a illustrates the comparison for zero initial conditions and Fig. 1b for impulsive disturbances. The agreement between the two responses is remarkable. With an increase in eccentricity and disturbances and a decrease in inertia ratio, the amplitude of the oscillations increases which correspondingly deteriorates the usefulness of the analytical solution, as shown in Fig. 1c.

Figure 2 is concerned with periodic motion at orbital frequency. In view of the assumption of linear oscillations, Fig. 2a shows the variation of initial conditions for a limited range of inertia and eccentricity. The results agree satisfactorily with those in Refs. 1 and 5. A convincing comparison is shown in Fig. 2b for two specific initial conditions.

IV. Concluding Remarks

The two-variable expansion procedure is successfully employed to predict the pitching librations of an arbitrary gravity-stabilized artificial rigid satellite in an eccentric orbit. In the linear range, a notable agreement between analytic and numerical responses is illustrated. For large amplitude motion, however, the solution may be useful only in a qualitative sense. This analysis also obtains a closed form relation between the system parameters and the initial conditions for the fundamental periodic solutions which play an important role in the stability investigation of the system.

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